

Hawking radiation of Dirac particles from black strings

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ABSTRACT: Hawking radiation has been studied as a phenomenon of quantum tunneling in different black holes. In this paper we extend this semi-classical approach to cylindrically symmetric black holes. Using the Hamilton-Jacobi method and WKB approximation we calculate the tunneling probabilities of incoming and outgoing Dirac particles from the event horizon and find the Hawking temperature of these black holes. We obtain results both for uncharged as well as charged particles.

1. Introduction

The semi-classical tunneling approach to Hawking radiation which was initially developed for the Schwarzschild black hole [1, 2, 3] has since then been successfully applied to a wide range of spherically symmetric black holes or spherical black holes with an axis of symmetry. These include the Kerr, Kerr-Newman, Taub-NUT, Gödel, dilatonic black holes and those with acceleration and rotation [4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14]. This has proved to be a quite robust technique mathematically. In this approach the rate of emission and absorption of particles across the black hole horizon is calculated from the imaginary part of their classical action [15, 16]. This method gives the Hawking temperature of black holes as well.

Another type of axial symmetry is cylindrical symmetry. Cylindrically symmetric fields are axisymmetric about an infinite axis (Killing vector, ∂_θ) and translationally symmetric along that axis (Killing vector, ∂_z). If we include time translation (∂_t) also then the stationary cylindrically symmetric spacetimes admit three Killing vectors, $\partial_t, \partial_\theta, \partial_z$ as the minimal symmetry with the algebra $\mathbb{R} \otimes SO(2) \otimes \mathbb{R}$. Ever since the first investigations of cylindrically symmetric black hole solutions [17, 18, 19, 20] with negative cosmological constant they have been a subject of interest for mathematical and physical properties. These solutions are asymptotically anti-de Sitter in transverse direction and along the axis. They have also been studied in the context of supergravity theories, topological defects and low energy string theories [21, 22, 23] and hence the name *black strings*. These objects have also been discussed in the presence of Born-Infeld and power Maxwell invariant fields, and in the context of higher dimensional and Gauss-Bonnet gravity theories, and for their thermodynamical properties [24, 25, 26, 27, 28].

In this paper we show that the tunneling method which has thus far been applied to spherical configurations can also be applied to cylindrical black holes giving results that are consistent with the literature. For this purpose we study radiation of uncharged and charged fermions from black strings. We find the tunneling probabilities of these particles using the WKB approximation and then calculate Hawking temperature at the horizon.

The solution of Einsteins field equations with a negative cosmological constant in the form of cylindrically symmetric spacetime when mass is the only parameter is given by [29]

$$ds^2 = - \left(\alpha^2 r^2 - \frac{M}{r} \right) dt^2 + \left(\alpha^2 r^2 - \frac{M}{r} \right)^{-1} dr^2 + r^2 d\theta^2 + \alpha^2 r^2 dz^2, \quad (1.1)$$

where $-\infty < t, z, < \infty$, $0 \leq r < \infty$, $0 \leq \theta \leq 2\pi$, $\alpha = -\Lambda/3$, Λ being the cosmological constant, and M is related to the ADM mass density of the black string. We write

$$F(r) = \left(\alpha^2 r^2 - \frac{M}{r} \right), \quad (1.2)$$

giving the event horizon of the black hole to be

$$r_+ = \left(\frac{M}{\alpha^2} \right)^{\frac{1}{3}}. \quad (1.3)$$

2. Tunneling of Dirac particles

Hawking radiation from black holes comprises different types of charged and uncharged particles. In this section we investigate tunneling of Dirac particles from the event horizon of black string solution via tunneling formalism. We assume that the massless spinor field Ψ obeys the general covariant Dirac's equation

$$\iota \gamma^\mu D_\mu \Psi + \frac{m}{\hbar} \Psi = 0, \quad (2.1)$$

where \hbar is the reduced Planck's constant, and

$$D_\mu = \partial_\mu + \Omega_\mu, \quad \Omega_\mu = \frac{1}{2} \iota \Gamma_\mu^{\alpha\beta} \Sigma_{\alpha\beta}, \quad \Sigma_{\alpha\beta} = \frac{1}{4} \iota [\gamma^\alpha, \gamma^\beta].$$

The γ^μ matrices satisfy $[\gamma^\alpha, \gamma^\beta] = -[\gamma^\beta, \gamma^\alpha]$, when $\alpha \neq \beta$ and $[\gamma^\alpha, \gamma^\beta] = 0$, when $\alpha = \beta$. Using this and values of the Christoffel symbols $\Gamma_\mu^{\alpha\beta}$, we note that $\Omega_\mu = 0$, thus yielding $D_\mu = \partial_\mu$. This reduces Dirac's equation to

$$\iota \gamma^\mu \partial_\mu \Psi + \frac{m}{\hbar} \Psi = 0. \quad (2.2)$$

We can choose the γ^μ matrices as

$$\gamma^t = \frac{1}{\sqrt{F(r)}} \gamma^0, \quad \gamma^r = \sqrt{F(r)} \gamma^3, \quad \gamma^\theta = \frac{1}{r} \gamma^1, \quad \gamma^z = \frac{1}{\alpha r} \gamma^2, \quad (2.3)$$

where

$$\begin{aligned} \gamma^0 &= \begin{pmatrix} \iota & 0 \\ 0 & -\iota \end{pmatrix}, \quad \gamma^1 = \begin{pmatrix} 0 & \sigma^1 \\ \sigma^1 & 0 \end{pmatrix}, \\ \gamma^2 &= \begin{pmatrix} 0 & \sigma^2 \\ \sigma^2 & 0 \end{pmatrix}, \quad \gamma^3 = \begin{pmatrix} 0 & \sigma^3 \\ \sigma^3 & 0 \end{pmatrix}. \end{aligned} \quad (2.4)$$

Here σ^μ are the Pauli matrices

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \sigma^2 = \begin{pmatrix} 0 & -\iota \\ \iota & 0 \end{pmatrix}, \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

In order to find the solution of Eq. (2.2) in the background of the black string (Eq. (1.1)) we employ the following ansatz for Dirac's field, corresponding to the spin up case

$$\Psi_\uparrow(t, r, \theta, z) = \begin{pmatrix} A(t, r, \theta, z) \xi_\uparrow \\ B(t, r, \theta, z) \xi_\uparrow \end{pmatrix} \exp\left(\frac{\iota}{\hbar} I\right), \quad (2.5)$$

where $\xi_\uparrow = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, I represents the classical action, and A and B are arbitrary functions of the coordinates. Using the above values we evaluate the four terms of Eq. (2.2) one by one and apply WKB approximation. We divide by the exponential term and multiply by \hbar . Considering terms only upto the leading order in \hbar , this procedure yields the following set of four equations.

$$\frac{-\iota A I_t}{\sqrt{F(r)}} - B \sqrt{F(r)} I_r + mA = 0, \quad (2.6)$$

$$-B \left(\frac{I_\theta}{r} + \frac{\iota I_z}{\alpha r} \right) = 0, \quad (2.7)$$

$$\frac{\iota B I_t}{\sqrt{F(r)}} - A \sqrt{F(r)} I_r + mB = 0, \quad (2.8)$$

$$-A \left(\frac{I_\theta}{r} + \frac{\iota I_z}{\alpha r} \right) = 0. \quad (2.9)$$

The derivatives involving higher order in \hbar have been neglected as we are taking only the lowest order in WKB approximation. Taking into account the Killing vectors of the background spacetime we can employ the following ansatz for the action in the spin up case

$$I_\uparrow(t, r, \theta, z) = -Et + l\theta + Jz + W(r), \quad (2.10)$$

where E is the energy of the emitted particles and W is the part of the action I_\uparrow that contributes to the tunneling probability. Using this ansatz in Eqs. (2.6)–(2.9) and noting that the contribution of J and l to the imaginary part of the action is canceled out we neglect the equations containing J and l . So, for the massless case, $m = 0$, the function $W(r)$ can be calculated only from the following equations

$$-\iota AE + BF(r) \frac{dW}{dr} = 0, \quad (2.11)$$

$$\iota BE + AF(r) \frac{dW}{dr} = 0. \quad (2.12)$$

In this case we get $A = -\iota B$ and

$$\left(\frac{dW_+}{dr} \right) = \frac{E}{F(r)}. \quad (2.13)$$

Similarly, the other equation yields $A = \iota B$ and

$$\left(\frac{dW_-}{dr} \right) = \frac{-E}{F(r)}. \quad (2.14)$$

In order to integrate Eq. (2.13) we write it as

$$W_+(r) = \int \frac{E}{F(r)} dr. \quad (2.15)$$

In this integral $r = r_+$ is a simple pole, therefore integrating around the pole we get

$$W_+(r) = \frac{\pi \iota E}{2\alpha^2 r_+ + \frac{M}{r_+^2}}. \quad (2.16)$$

Similarly

$$W_-(r) = \frac{-\pi \iota E}{2\alpha^2 r_+ + \frac{M}{r_+^2}}. \quad (2.17)$$

The probabilities of crossing the horizon in each direction can be given by [15, 16]

$$P_{\text{emission}} \propto \exp(-2ImI) = \exp(-2ImW_+),$$

$$P_{\text{absorption}} \propto \exp(-2ImI) = \exp(-2ImW_-).$$

While computing the imaginary part of the action, we note that it is same for both the incoming and outgoing solutions. Now the probability of particles tunneling from inside to outside the horizon is given by

$$\Gamma \propto \frac{P_{\text{emission}}}{P_{\text{absorption}}} = \frac{\exp(-2ImW_+)}{\exp(-2ImW_-)},$$

which gives

$$\Gamma = \exp(-4ImW_+). \quad (2.18)$$

Using the value of W_+ this becomes

$$\Gamma = \exp\left(\frac{-4\pi E}{2\alpha^2 r_+ + \frac{M}{r_+^2}}\right). \quad (2.19)$$

Now we know that the tunneling probability is given by $\Gamma = \exp(-\beta E)$, where $\beta = \frac{1}{T_H}$, yielding

$$T_H = \frac{1}{4\pi} \left(2\alpha^2 r_+ + \frac{M}{r_+^2} \right), \quad (2.20)$$

which is the correct Hawking temperature for black strings [28, 29]. For the massive case (i.e. $m \neq 0$) we get

$$\left(\frac{A}{B} \right)^2 = \frac{-\iota E + m\sqrt{F(r)}}{\iota E + m\sqrt{F(r)}}.$$

Near the horizon we get $A^2 = -B^2$, and obtain the same Hawking temperature as in the massless case. This is because near the horizon the massive particles behave like massless particles.

3. Tunneling from the charged black string

Soon after the discovery of the cylindrically symmetric black hole solution it was extended to the case with electromagnetic field [29]. Here we study quantum tunneling of Dirac particles from charged black strings. The line element can be written as

$$ds^2 = - \left(\alpha^2 r^2 - \frac{4M}{\alpha r} + \frac{4Q^2}{\alpha^2 r^2} \right) dt^2 + \left(\alpha^2 r^2 - \frac{4M}{\alpha r} + \frac{4Q^2}{\alpha^2 r^2} \right)^{-1} dr^2 + r^2 d\theta^2 + \alpha^2 r^2 dz^2, \quad (3.1)$$

with the scalar potential

$$F_{tr} = \frac{-2Q}{\alpha r}. \quad (3.2)$$

Here M is mass, Q is charge, $\alpha = -\Lambda/3$. Putting $g^{11} = 0$ gives the location of the inner and outer horizons [30] for the non-extremal charged black string as

$$r_{\pm} = \frac{1}{2} \left[\sqrt{2R} \pm \left(-2R + \frac{8M}{\alpha^3 \sqrt{2R}} \right)^{\frac{1}{2}} \right], \quad (3.3)$$

where

$$R = \left\{ \frac{M^2}{\alpha^6} + \left[\left(\frac{M^2}{\alpha^6} \right)^2 - \left(\frac{4Q^2}{3\alpha^4} \right)^3 \right]^{\frac{1}{2}} \right\}^{\frac{1}{3}} + \left\{ \frac{M^2}{\alpha^6} - \left[\left(\frac{M^2}{\alpha^6} \right)^2 - \left(\frac{4Q^2}{3\alpha^4} \right)^3 \right]^{\frac{1}{2}} \right\}^{\frac{1}{3}}. \quad (3.4)$$

In this case the function $F(r)$ takes the following form

$$F(r) = \frac{\alpha^4 r^4 - 4M\alpha r + 4Q^2}{\alpha^2 r^2}, \quad (3.5)$$

The charged Dirac equation for a particle with charge q is given by

$$\iota\gamma^\mu \left(D_\mu - \frac{\iota q}{\hbar} A_\mu \right) \Psi + \frac{m}{\hbar} \Psi = 0. \quad (3.6)$$

We choose the γ^μ matrices as before but noting that the function F is now given by Eq. (3.5). Assuming the Dirac field of the form taken earlier and applying WKB approximation we obtain the following set of four equations

$$-\iota A \frac{I_t}{\sqrt{F(r)}} - B \sqrt{F(r)} I_r - 2\iota A \frac{Qq}{\alpha r \sqrt{F(r)}} + mA = 0, \quad (3.7)$$

$$B \left(\frac{I_\theta}{r} + \frac{\iota I_z}{\alpha r} \right) = 0, \quad (3.8)$$

$$\iota B \frac{I_t}{\sqrt{F(r)}} - A \sqrt{F(r)} I_r + 2\iota B \frac{Qq}{\alpha r \sqrt{F(r)}} + mB = 0, \quad (3.9)$$

$$A \left(\frac{I_\theta}{r} + \frac{\iota I_z}{\alpha r} \right) = 0. \quad (3.10)$$

Substituting the same form of the action as earlier and following an analysis similar the one done in the uncharged case we find that

$$A = -\iota B \quad \text{and} \quad \frac{dW}{dr} = \frac{E\alpha r - 2Qq}{\alpha r F(r)}. \quad (3.11)$$

Using the value of $F(r)$ in Eq. (3.11) and integrating by employing the residue theory we get

$$W_+ = \pi\iota \left(\frac{E\alpha^2 r_+^3 - 2Q\alpha r_+^2 q}{2\alpha^4 r_+^4 + 4M\alpha r_+ - 8Q^2} \right). \quad (3.12)$$

Also

$$W_- = \pi\iota \left(\frac{-E\alpha^2 r_+^3 + 2Q\alpha r_+^2 q}{2\alpha^4 r_+^4 + 4M\alpha r_+ - 8Q^2} \right).$$

Now, as before we write the probability of particles tunneling from inside to outside the horizon as

$$\Gamma = \exp \left[\frac{-4\pi (E\alpha^2 r_+^3 - 2Q\alpha r_+^2 q)}{2\alpha^4 r_+^4 + 4M\alpha r_+ - 8Q^2} \right]. \quad (3.13)$$

Comparing this with $\Gamma = \exp(-E/T_H)$, the Hawking temperature for charged black string takes the form

$$T_H = \frac{1}{2\pi} \left(\alpha^2 r_+ + \frac{2M}{\alpha r_+^2} - \frac{4Q^2}{\alpha^2 r_+^3} \right). \quad (3.14)$$

which is consistent with the literature [28, 29].

4. Conclusion

In the tunneling formalism the probability of particles crossing the black hole horizon on either sides are calculated using complex path integrals. The particles traverse geodesics which are forbidden in classical treatments. However the probability for absorption of particles should actually be equal to 1 as this is a path which is permitted classically [1]. This provides an efficient way of computing the Hawking temperature as well. This technique has been successfully used for spherically symmetric black holes. We have extended its application to cylindrically symmetric configurations. Solving Dirac's equations in the background of uncharged and charged black strings and applying WKB approximation, we have calculated the tunneling probability of fermions. Hawking temperature in both the cases is also calculated.

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